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Similarity transformations in one- and two-mode Fock space

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Abstract. The normally ordered Hilbert space image of the general (complex) linear similarity transformation in phase space is obtained in coherent state representation. Although preserving the commutator of a and $a^\dagger$, these quantum mechanical images of classical transformations are generally not unitary. Remarkably, although the kets and bras produced by the non-unitary similarity transformations are not Hermitian conjugates, squeezed state analogues produced using the similarity transformation still satisfy an overcompleteness relation. The results are extended to two-mode Fock space and simple examples of the utility of the similarity transformation are presented. The evaluation of the normally ordered operators is greatly facilitated by the use of *integration within ordered products*.

1. Introduction

Within quantum mechanics a change of basis or representation is effected by means of a unitary transformation, U, where the bra $|\alpha\rangle$ in the new basis is $U|\alpha\rangle$ and the corresponding ket $\langle\alpha|U^\dagger$ [1]. The (non-unitary) complex linear transformation of canonical variables has been considered by Wolf [2] and Kramer *et al* [3]. The mapping of the position and momentum operators onto the annihilation operator a and creation operator $a^\dagger$ is a familiar example of such a non-unitary transformation.

It is well known that unitary transformations preserve completeness of a basis. For coherent states $|\alpha\rangle = D(\alpha)|0\rangle$, this completeness is expressed by the relation [4]

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1 \tag{1.1}$$

where $d^2\alpha \equiv d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha)$. It occurred to us that closure might not be limited to the Hermitian conjugate bra and ket pair produced by unitary transformations, but might in fact extend to states transformed by the more general similarity transformation such that $|\varphi\rangle = D(\alpha)W|0\rangle$ and $\langle\xi| = \langle 0|W^{-1}D^\dagger(\alpha)$ generate the relation

$$\int \frac{d^2\alpha}{\pi} |\varphi\rangle\langle\xi| = 1. \tag{1.2}$$

Thus, although $|\varphi\rangle$ and $\langle\xi|$ are not Hermitian conjugates, they still form an (over)complete basis.

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In this paper we obtain the coherent state representation and the normally ordered form of the general linear similarity transformation of $a^\dagger$ and a, the creation and annihilation operators of the harmonic oscillator. Transforming the ground state of the harmonic oscillator with this similarity transformation, followed by a Glauber displacement, we produce the eigenstates of the non-Hermitian images of $a^\dagger$ and a and, with one taken as bra and the other as ket, we show that these eigenstates satisfy the completeness relation (1.2).

In this introduction we briefly recapitulate an earlier paper [5] in order to establish the method of proceeding and to define our terms. In section 2 we obtain the transformation required in one-mode Fock space and extend this result to two-mode Fock space in section 3. The completeness of the squeezed state analogues is demonstrated in section 4 and finally in section 5 we give several applications of the completeness relation and the similarity transformations.

In [5] we investigated how real classical linear transformations in coordinate momentum phase space are mapped to unitary quantum mechanical operators in Hilbert space. The resulting operators were evaluated using the *integration within ordered products* (IWOP) [6] technique in the coherent state representation, and were shown to be a generalization of the familiar squeeze operators [7]. In this paper we will further generalize the transformations to encompass complex canonical transformations. To briefly review the results of [5], it was shown that the operator U defined by

$$U(g)=(2\pi)^{-1}s^{-1/2}|s|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathrm{d}q\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\ p\end{pmatrix}\right| \tag{1.3}$$

where

$$\left|\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle=\exp\{-\tfrac{1}{4}(p^2+q^2)+2^{-1/2}(q+\mathrm{i}p)a^\dagger\}|0\rangle \tag{1.4}$$

is the coherent state, $|0\rangle$ the harmonic oscillator's ground state, $a^\dagger=2^{-1/2}(\hat{Q}-\mathrm{i}\hat{P})$ its creation operator (for convenience we have taken $\hbar=\omega=m=1$) and

$$s=\tfrac{1}{2}[(A+D)+\mathrm{i}(B-C)] \tag{1.5}$$

generates the transformation

$$U^\dagger\hat{Q}U=A\hat{Q}+B\hat{P}\qquad U^\dagger\hat{P}U=C\hat{Q}+D\hat{P} \tag{1.6}$$

with A, B, C, D real and $AD-BC=1$. Rewriting the ket in (1.3) as

$$\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle=\left|\begin{pmatrix}s^* & -r\\ -r^* & s\end{pmatrix}\begin{pmatrix}z\\ z^*\end{pmatrix}\right\rangle\equiv|s^*z-rz^*\rangle \tag{1.7}$$

with $z=2^{-1/2}(q+\mathrm{i}p)$ and $r=\frac{1}{2}[(D-A)-\mathrm{i}(B+C)]$, we recast (1.3) into the form

$$U(g)=s^{-1/2}|s|\int\pi^{-1}\,\mathrm{d}^2z|s^*z-rz^*\rangle\langle z| \tag{1.8}$$

with

$$|z\rangle=\mathrm{e}^{za^\dagger-z^*a}|0\rangle\equiv|z\rangle. \tag{1.9}$$

The coherent state $|z\rangle$ satisfies $a|z\rangle=z|z\rangle$. Expression (1.8) was then integrated using the IWOP technique. Equation (1.6) may be rewritten as

$$a'=U^\dagger aU=s^*a-ra^\dagger\qquad a^{\dagger\prime}=U^\dagger a^\dagger U=sa^\dagger-ra^* \tag{1.10}$$

with $|s|^2-|r|^2=1$. Clearly a' and $a^{\dagger\prime}$ are Hermitian conjugates and $[a', a^{\dagger\prime}]=1$. In passing, we note also that $U^\dagger = U^{-1}$.

In section 2 we investigate the general linear similarity transformation of a and $a^\dagger$ given by

$$d = WaW^{-1} = \mu a + \nu a^\dagger \qquad g^\dagger = Wa^\dagger W^{-1} = \sigma a + \tau a^\dagger \tag{1.11}$$

with μ, ν, σ, τ arbitrary complex numbers satisfying $\mu\tau - \sigma\nu = 1$. It is easily seen that the similarity transformation W preserves the commutator $[d, g^\dagger]=1$ even though d and $g^\dagger$ are not generally Hermitian conjugates.

2. Derivation of W in the coherent state representation

Guided by the earlier work quoted in (1.7), we postulate the following coherent state representation for W:

$$W = \tau^{1/2} \int \pi^{-1}\, \mathrm{d}^2 z \left| \begin{pmatrix} \tau & -\nu \\ -\sigma & \mu \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} z \\ z^* \end{pmatrix} \right| \tag{2.1}$$

with $\mu\tau - \sigma\nu = 1$, where we make, consistent with (1.9), the definition

$$\left| \begin{pmatrix} \tau & -\nu \\ -\sigma & \mu \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \right\rangle = \left| \begin{pmatrix} \tau z - \nu z^* \\ \mu z^* - \sigma z \end{pmatrix} \right\rangle \equiv \mathrm{e}^{(\tau z - \nu z^*)a^\dagger - (\mu z^* - \sigma z)a} |0\rangle. \tag{2.2}$$

Note that in contrast to expression (1.9) for the usual coherent state, the coefficients of a and $a^\dagger$ are not complex conjugates.

We expand (2.2) using the Baker–Hausdorff formula to obtain

$$(2.2) = \exp[(\tfrac{1}{2} - \mu\tau)|z|^2 + (z\tau - z^*\nu)a^\dagger + \tfrac{1}{2}\nu\mu z^{*2} + \tfrac{1}{2}\sigma\tau z^2]|0\rangle. \tag{2.3}$$

With the aid of a formula from [6],

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp(\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) = (\zeta^2 - 4fg)^{-1/2} \exp\left(\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg} \right) \tag{2.4}$$

having convergence conditions

$$\begin{cases} \mathrm{Re}(\zeta + f + g) < 0 \\ \mathrm{Re}[(\zeta^2 - 4fg)/(\zeta + f + g)] < 0 \end{cases} \quad \text{or} \quad \begin{cases} \mathrm{Re}(\zeta - f - g) < 0 \\ \mathrm{Re}[(\zeta^2 - 4fg)/(\zeta - f - g)] < 0 \end{cases}$$

we effect the integration of (2.1) using the IWOP technique:

$$\begin{aligned} W &= \tau^{1/2} \int \frac{\mathrm{d}^2 z}{\pi} :\exp\left(-\mu\tau|z|^2 + z\tau a^\dagger + z^*(a - \nu a^\dagger) + \frac{\mu\nu z^{*2}}{2} + \frac{\sigma\tau z^2}{2} - a^\dagger a \right): \\ &= \mu^{-1/2} :\exp\left(\frac{-\nu}{2\mu} a^{\dagger 2} \right) \exp[(\mu^{-1} - 1)a^\dagger a] \exp\left(\frac{\sigma}{2\mu} a^2 \right): \\ &= \mu^{-1/2} \exp\left(\frac{-\nu}{2\mu} a^{\dagger 2} \right) \exp(-a^\dagger a \ln \mu) \exp\left(\frac{\sigma}{2\mu} a^2 \right) \end{aligned} \tag{2.5}$$

where : : denotes the normal product and we have used $|0\rangle\langle 0| = :\mathrm{e}^{-a^\dagger a}:$ and $\mathrm{e}^{\lambda a^\dagger a} = :\exp\{(\mathrm{e}^\lambda - 1)a^\dagger a\}:$.

With the aid of (2.5) and its inverse

$$W^{-1} = \mu^{1/2} \exp\left(\frac{-\sigma}{2\mu} a^2\right) \exp(a^\dagger a \ln \mu) \exp\left(\frac{\nu}{2\mu} a^{\dagger 2}\right) \tag{2.6}$$

we easily verify that W generates the transformations (1.10) as

$$d = WaW^{-1} = \exp\left(\frac{-\nu}{2\mu} a^{\dagger 2}\right)\mu a \exp\left(\frac{\nu}{2\mu} a^{\dagger 2}\right) = \mu a + \nu a^\dagger \tag{2.7a}$$

$$g^\dagger = Wa^\dagger W^{-1} = \exp\left(\frac{-\nu}{2\mu} a^{\dagger 2}\right)\left(\frac{a^\dagger}{\mu} + \sigma a\right) \exp\left(\frac{\nu}{2\mu} a^{\dagger 2}\right) = \tau a^\dagger + \sigma a. \tag{2.7b}$$

Having obtained the normally ordered form of W, we may forget its origins. In particular we lift the convergence restrictions of the integral (2.4) above. The inverse transformations follow immediately from (2.6) and (2.7):

$$W^{-1} a W = \tau a - \nu a^\dagger \qquad \text{and} \qquad W^{-1} a^\dagger W = \mu a^\dagger - \sigma a. \tag{2.8}$$

We will also require W^{-1} in *normal* order. Again using IWOP, (2.4) and the overcompleteness relation for the coherent state,

$$\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\langle z| = \int \frac{\mathrm{d}^2 z}{\pi} :\exp(-|z|^2 + za^\dagger + z^* a - a^\dagger a): = 1 \tag{2.9}$$

we re-evaluate

$$\begin{aligned} W^{-1} &= \mu^{1/2} \exp\left(\frac{-\sigma}{2\mu} a^2\right) \int \frac{\mathrm{d}^2 z}{\pi} \exp\left(-\frac{|z|^2}{2} + \mu z a^\dagger\right)|0\rangle\langle z| \exp\left(\frac{\nu}{2\mu} z^{*2}\right) \\ &= \mu^{1/2} \int \frac{\mathrm{d}^2 z}{\pi} :\exp\left(-|z|^2 + \mu z a^\dagger - \frac{\sigma\mu}{2} z^2 + z^* a + \frac{\nu}{2\mu} z^{*2} - a^\dagger a\right): \\ &= \tau^{-1/2} \exp\left(\frac{\nu a^{\dagger 2}}{2\tau}\right) \exp(-a^\dagger a \ln \tau) \exp\left(\frac{-\sigma}{2\tau} a^2\right) \neq W^\dagger. \end{aligned} \tag{2.10}$$

The conditions of (2.4) may in fact be shown to be overly restrictive; an explicit evaluation of WW^{-1} gives $WW^{-1} = W^{-1}W = 1$.

The classical phase space map analogous to that of (1.6) may now be constructed. Setting

$$\begin{aligned} W\hat{Q}W^{-1} &= 2^{-1/2} W(a + a^\dagger) W^{-1} = 2^{-1/2}(\mu a + \nu a^\dagger + \sigma a + \tau a^\dagger) \\ &= \tfrac{1}{2}[\hat{Q}(\mu + \nu + \sigma + \tau) + \mathrm{i}P(\mu + \sigma - \nu - \tau)]. \end{aligned}$$

Similarly, $WPW^{-1} = \frac{1}{2}[\hat{P}(\mu - \sigma + \tau - \nu) + \mathrm{i}\hat{Q}(\sigma - \mu + \tau - \nu)]$. Thus W is the Hilbert space image of the general complex phase space map

$$q' = A'q + B'p \qquad p' = C'q + D'p \tag{2.11}$$

with

$$\begin{aligned} A' &= \tfrac{1}{2}(\mu + \nu + \sigma + \tau) \qquad & B' &= \frac{\mathrm{i}}{2}(\mu + \sigma - \nu - \tau) \\ C' &= \frac{\mathrm{i}}{2}(\sigma + \tau - \nu - \mu) \qquad & D' &= \tfrac{1}{2}(\mu - \sigma - \nu + \tau). \end{aligned} \tag{2.12}$$

The coefficients satisfy $A'D'-B'C'=1$ and (2.1) can be cast in the form

$$W=\tau^{1/2}\int\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi}\left|\begin{pmatrix}D' & -B'\\ -C' & A'\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\ p\end{pmatrix}\right|. \tag{2.13}$$

Conversely, given the phase space map above, μ, ν, σ, τ may be found as

$$\begin{aligned}\mu&=\tfrac{1}{2}[(A'+D')+\mathrm{i}(C'-B')] &\qquad \nu&=\tfrac{1}{2}[(A'-D')+\mathrm{i}(B'+C')]\\ \sigma&=\tfrac{1}{2}[(A'-D')-\mathrm{i}(B'+C')] &\qquad \tau&=\tfrac{1}{2}[(A'+D')+\mathrm{i}(B'-C')].\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.7) we obtain

$$W\hat{Q}W^{-1}=\hat{A}'Q+B'\hat{P} \qquad \text{and} \qquad W\hat{P}W^{-1}=C'\hat{Q}+D'\hat{P}. \tag{2.15}$$

The commutator $[A'\hat{Q}+B'\hat{P},\ C'\hat{Q}+D'\hat{P}]=1$, and we note that W is not a unitary transformation unless A', B', C' and D' are all real.

3. The two-mode transformation

We next turn our attention to linear maps in the space generated by two harmonic oscillators. We let a be the annihilation operator of the first mode whose canonical coordinates are q_1 and p_1, and b the annihilation operator for the second mode with coordinates q_2 and p_2. We define, as for the one-mode case, $z_1\equiv 2^{-1/2}(q_1+\mathrm{i}p_1)$ and $z_2\equiv 2^{-1/2}(q_2+\mathrm{i}p_2)$. In [5] we obtained the unitary transformation $U^{(2)}$ that generalizes the transformation of a and b produced by the customary two-mode squeeze operator [8]

$$\begin{aligned}SaS^{\dagger}&=a\cosh r+b^{\dagger}\exp(2\mathrm{i}\varphi)\sinh r\\ SbS^{\dagger}&=b\cosh r+a^{\dagger}\exp\{2\mathrm{i}\varphi)\sinh r\end{aligned} \tag{3.1}$$

to

$$\begin{aligned}a''&=U^{(2)}aU^{(2)\dagger}=s^*a-rb^{\dagger} &\qquad b''&=U^{(2)}bU^{(2)\dagger}=s^*b-ra^{\dagger}\\ a^{\dagger\prime\prime}&=U^{(2)}a^{\dagger}U^{(2)\dagger}=sa^{\dagger}-r^*b &\qquad b^{\dagger\prime\prime}&=U^{(2)}b^{\dagger}U^{(2)\dagger}=sb^{\dagger}-r^*a.\end{aligned} \tag{3.2}$$

We now seek to further generalize this result by finding the similarity transformation V that maintains the commutation relations for the transformed annihilation and creation operators, without requiring unitarity. In other words, we seek V that transforms a, $a^{\dagger}$, b and $b^{\dagger}$ according to

$$\begin{aligned}VaV^{-1}&=\mu a+\nu b^{\dagger} &\qquad VbV^{-1}&=\mu b+\nu a^{\dagger}\\ Va^{\dagger}V^{-1}&=\tau a^{\dagger}+\sigma b &\qquad Vb^{\dagger}V^{-1}&=\tau b^{\dagger}+\sigma a.\end{aligned} \tag{3.3}$$

Guided by the results of [5], we tentatively identify V as

$$V=\tau\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}\left|\begin{pmatrix}\tau & 0 & 0 & -\nu\\ 0 & \mu & -\sigma & 0\\ 0 & -\nu & \tau & 0\\ -\sigma & 0 & 0 & \mu\end{pmatrix}\begin{pmatrix}z_1\\ z_1^*\\ z_2\\ z_2^*\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z_1\\ z_1^*\\ z_2\\ z_2^*\end{pmatrix}\right|. \tag{3.4}$$

Following the example of (2.2), we express this as

$$\begin{aligned}V=\tau\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}&\exp[-\tfrac{1}{2}(z_1\tau-z_2^*\nu)(z_1^*\mu-z_2\sigma)-\tfrac{1}{2}(z_2\tau-z_1^*\nu)(z_2^*\mu-z_1\sigma)\\ &+(z_1\tau-z_2^*\nu)a^{\dagger}+(z_2\tau-z_1^*\nu)b^{\dagger}]|00\rangle\langle z_1z_2|\end{aligned} \tag{3.5}$$

which may be recast as

$$V=\tau\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}:\exp[-\mu\tau(|z_1|^2+|z_2|^2)+z_1(a^\dagger+z_2\sigma)\tau+z_1^*(-\nu b^\dagger+a+z_2^*\mu\nu)$$
$$+z_2\tau b^\dagger+z_2^*(b-\nu a^\dagger)-aa^\dagger-bb^\dagger]: \tag{3.6}$$

and integrated (when $\mathrm{Re}(\mu\tau)>0$) using IWOP to obtain

$$\begin{aligned}V&=\mu^{-1}:\exp[-\nu\mu^{-1}a^\dagger b^\dagger+(a^\dagger a+b^\dagger b)(\mu^{-1}-1)+(\sigma\mu^{-1}ab)]:\\&=\mu^{-1}\exp(-\nu\mu^{-1}a^\dagger b^\dagger)\exp[-(a^\dagger a+b^\dagger b)\ln\mu]\exp(\sigma\mu^{-1}ab).\end{aligned}\tag{3.7}$$

It is now a simple matter to verify that (3.7) is indeed the operator required to effect the similarity transformation (3.3). Proceeding as in (2.11), we find that V is the Hilbert space image of the two-mode phase space map

$$\begin{pmatrix}q_1'\\q_2'\\p_1'\\p_2'\end{pmatrix}=\frac{1}{2}\begin{pmatrix}A'+D' & -A'+D' & -B'+C' & -B'-C'\\ -A'+D' & A'+D' & -B'-C' & -B'+C'\\ -C'+B' & -B'-C' & A'+D' & -D'+A'\\ -B'-C' & -C'+B' & -D'+A' & A'+D'\end{pmatrix}\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}. \tag{3.8}$$

As in the single-mode case we shall require V^{-1} in normally ordered form. From (3.7)

$$\begin{aligned}V^{-1}&=\mu\exp\left(-\frac{\sigma}{\mu}ab\right)\exp[(a^\dagger a+b^\dagger b)\ln\mu]\exp\left(\frac{\nu}{\mu}a^\dagger b^\dagger\right)\\
&=\mu\exp\left(-\frac{\sigma}{\mu}ab\right)\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}\\
&\quad\times\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)+z_1\mu a^\dagger+z_2\mu b^\dagger]|00\rangle\langle z_1z_2|\exp\left(\frac{\nu}{\mu}z_1^*z_2^*\right)\\
&=\mu\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}:\exp(-|z_1|^2-|z_2|^2+z_1\mu a^\dagger+z_1\mu b^\dagger-\sigma\mu z_1z_2+z_1^*a\\
&\quad+z_2^*b+\nu\mu^{-1}z_1^*z_2^*-a^\dagger a-b^\dagger b):\\
&=\tau^{-1}\exp\left(\frac{\nu}{\tau}a^\dagger b^\dagger\right)\exp[-(b^\dagger b+a^\dagger a)\ln\tau]\exp\left(-\frac{\sigma}{\tau}ab\right)\neq V^\dagger.\end{aligned}\tag{3.9}$$

4. Transformed coherent states

4.1. Single mode

Traditionally [9, 10] *squeezed* states may be generated by squeezing the ground state $|0\rangle$ and then displacing the resulting squeezed ground state $|0\rangle_s$ with the Glauber displacement operator $D(\alpha)\equiv\exp(\alpha a^\dagger-\alpha^*a)$. Following this approach, we replace the traditional squeeze operator with W from (2.5). Thus the state $|\alpha;\mu,\nu\rangle\equiv D(\alpha)W|0\rangle$ is given by

$$D(\alpha)W|0\rangle=\mu^{-1/2}\exp\left(\frac{-\nu}{2\mu}(a^\dagger-\alpha^*)^2+\alpha a^\dagger-\tfrac{1}{2}|\alpha|^2\right)|0\rangle. \tag{4.1}$$

It is worth noting that $|\alpha;\mu,\nu\rangle$ satisfies the eigenvalue equation

$$(\mu a+\nu a^\dagger)|\alpha;\mu,\nu\rangle=(\mu\alpha+\nu\alpha^*)|\alpha;\mu,\nu\rangle. \tag{4.2}$$

In a similar fashion we construct $\langle\alpha;\tau,\sigma| \equiv \langle 0|W^{-1}D^{\dagger}(\alpha)$, noting that as W is not unitary W^{-1}, not $W^{\dagger}$, is required to produce the bra

$$\langle\alpha;\tau,\sigma| = \tau^{-1/2}\langle 0|\exp\left(\frac{-\sigma}{2\tau}(a-\alpha)^2+\alpha^* a-\tfrac{1}{2}|\alpha|^2\right) \tag{4.3}$$

which satisfies

$$\langle\alpha;\tau,\sigma|(\alpha a+\tau a^{\dagger}) = (\sigma\alpha+\tau\alpha^*)\langle\alpha;\tau,\sigma|. \tag{4.4}$$

In spite of the fact that $|\alpha;\mu,\nu\rangle$ and $\langle\alpha;\tau,\sigma|$ are not Hermitian conjugates, they do in fact satisfy the completeness relation (1.2). The completeness may be demonstrated (assuming the convergence of the integrals) with the aid of IWOP:

$$\begin{aligned}
\int\frac{d^2\alpha}{\pi}&|\alpha;\mu,\nu\rangle\langle\alpha;\tau,\sigma| \\
&= (\mu\tau)^{-1/2}\int\frac{d^2\alpha}{\pi}:\exp\bigg[-|\alpha|^2+\alpha\left(a^{\dagger}+\frac{\sigma}{\tau}a\right)+\alpha^*\left(a+\frac{\nu}{\mu}a^{\dagger}\right) \\
&\quad -\frac{\nu}{2\mu}(\alpha^{*2}+a^{\dagger 2})-\frac{\sigma}{2\tau}(\alpha^2+a^2)-a^{\dagger}a\bigg]: \\
&= :\exp\bigg[\left(\frac{\tau}{\sigma}a+a^{\dagger}\right)\left(a+\frac{\nu}{\mu}a^{\dagger}\right)-\frac{\nu}{2\mu}\left(a^{\dagger}+\frac{\sigma}{\tau}a\right)^2 \\
&\quad -\frac{\sigma}{2\tau}\left(a+\frac{\nu}{\mu}a^{\dagger}\right)^2-a^{\dagger}a-\frac{\nu}{2\mu}a^{\dagger 2}-\frac{\sigma}{2\tau}a^2\bigg]: \\
&= 1.
\end{aligned} \tag{4.5}$$

Thus we find that in contrast to unitary transformations where kets and their Hermitian conjugate bras satisfy the completeness relation, similarity transformations require bras to be acted on by W^{-1} rather than $W^{\dagger}$ to satisfy (1.1). In fact, $\int\pi^{-1}d^2\alpha|\alpha;\mu,\nu\rangle\langle\alpha;\mu,\nu| = (|\mu|^2-|\nu|^2)^{-1/2}$; only when $\tau=\mu^*$ and $\sigma=\nu^*$ does this equal 1, in which case W is unitary.

The alternative squeezed state analogues obtained by first displacing the ground state and then transforming

$$\begin{aligned}
|\alpha;\mu,\nu,\sigma\rangle &\equiv WD(\alpha)|0\rangle \\
&= \mu^{-1/2}\exp\left(-\frac{|\alpha|^2}{2}+\frac{\sigma\alpha^2}{2\mu}-\frac{\nu a^{\dagger 2}}{2\mu}-+\frac{\alpha a^{\dagger}}{\mu}\right)|0\rangle
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\langle\alpha;\tau,\nu,\sigma| &\equiv \langle 0|D^{\dagger}(\alpha)W^{-1} \\
&= \tau^{-1/2}\langle 0|\exp\left(-\frac{|\alpha|^2}{2}+\frac{\nu\alpha^{*2}}{2\tau}-\frac{\sigma a^2}{2\tau}+\frac{\alpha^* a}{\tau}\right)
\end{aligned} \tag{4.7}$$

satisfy the eigenvalue equations

$$(\mu a+\nu a^{\dagger})|\alpha;\mu,\nu,\sigma\rangle = \alpha|\alpha;\mu,\nu,\sigma\rangle$$

and

$$\langle\alpha;\tau,\nu,\sigma|(\tau a^{\dagger}+\sigma a) = \alpha^*\langle\alpha;\tau,\nu,\sigma|$$

as well as the relation

$$\int \pi^{-1}\, d^2\alpha\, |\alpha;\mu,\nu,\sigma\rangle\langle\alpha;\tau,\nu,\sigma| = 1.$$

4.2. Two mode

Proceeding as in the single-mode case, we obtain the two-mode squeezed state analogues

$$|\alpha\beta;\mu,\nu\rangle \equiv D(\alpha)D(\beta)V|00\rangle = \mu^{-1}\exp\left(-\frac{\mu}{\nu}(a^\dagger-\alpha^*)(b^\dagger-\beta^*)\right)|\alpha\beta\rangle \tag{4.8}$$

and

$$\langle\alpha\beta;\tau,\sigma| \equiv \langle 00|V^{-1}D^\dagger(\alpha)D^\dagger(\beta) = \tau^{-1}\langle\alpha\beta|\exp\left(-\frac{\sigma}{\tau}(a-\alpha)(b-\beta)\right). \tag{4.9}$$

The ket (4.8) and bra (4.9) satisfy the eigenvalue equations

$$(\mu a + \nu b^\dagger)|\alpha\beta;\mu,\nu\rangle = (\mu\alpha+\nu\beta^*)|\alpha\beta;\mu,\nu\rangle$$

and

$$(\mu b + \nu a^\dagger)|\alpha\beta;\mu,\nu\rangle = (\mu\beta+\nu\alpha^*)|\alpha\beta;\mu,\nu\rangle$$

$$\langle\alpha\beta;\tau,\sigma|(\tau a^\dagger + \sigma b) = (\tau\alpha^* + \alpha\beta)\langle\alpha\beta;\tau,\sigma|$$

and

$$\langle\alpha\beta;\tau,\sigma|(\tau b^\dagger + \sigma a) = (\tau\beta^* + \sigma\alpha)\langle\alpha\beta;\tau,\sigma|$$

and their completeness relation is verified below:

$$\begin{aligned}
\int \frac{d^2\alpha\, d^2\beta}{\pi^2}&|\alpha\beta;\mu,\nu\rangle\langle\alpha\beta;\tau,\sigma| \\
&= (\mu\tau)^{-1}\int \frac{d^2\alpha\, d^2\beta}{\pi} :\exp\left[-|\alpha|^2-|\beta|^2+\alpha\left(a^\dagger+\frac{\sigma}{\tau}b\right)\right. \\
&\quad +\alpha^*\left(a+\frac{\nu}{\mu}b^\dagger\right)+\beta\left(b^\dagger+\frac{\sigma}{\tau}a\right) \\
&\quad \left.+\beta^*\left(b+\frac{\nu}{\mu}a^\dagger\right)-\frac{\nu}{\mu}(\alpha^*\beta^*+a^\dagger b^\dagger)-\frac{\sigma}{\tau}(\alpha\beta+ab)-a^\dagger a-b^\dagger b\right]: \\
&= 1 \qquad (\text{provided } \mathrm{Re}(\mu\tau)>0).
\end{aligned} \tag{4.10}$$

Again we find the bras and kets related by similarity transformations satisfy the completeness relation while the Hermitian conjugate bras and kets do not.

5. Applications

In the following applications it is assumed that the conditions for convergence of the various integrals are met. As a first application we use the completeness relation (4.5) and the eigenvalue equation (4.2) to derive the normal product form of $\exp[\lambda(\mu a + \nu a^\dagger)^2]$:

$$\mathrm{e}^{\lambda(\mu a+\nu a^\dagger)^2} = \int \frac{\mathrm{d}^2\alpha}{\pi}\, \mathrm{e}^{\lambda(\mu\alpha+\nu\alpha^*)}|\alpha;\mu,\nu\rangle\langle\alpha;\tau,\sigma| \tag{5.1}$$

$$\begin{aligned} &= (\mu\tau)^{-1/2}\int\frac{\mathrm{d}^2\alpha}{\pi} :\exp\left[-|\alpha|^2+\alpha\left(a^\dagger+\frac{\sigma}{\tau}a\right)+\alpha^*\left(a+\frac{\nu}{\mu}a^\dagger\right)\right. \\ &\quad \left. -\frac{\nu}{2\mu}(a^{\dagger 2}+\alpha^{*2})-\frac{\alpha}{2\tau}(\alpha^2+a^2)-\lambda(\mu\alpha+\nu\alpha^*)^2-a^\dagger a\right]: \\ &= (1-2\mu\nu\lambda)^{-1/2}\exp\left(\frac{\lambda\nu^2 a^{\dagger 2}}{1-2\lambda\mu}\right)\exp[-a^\dagger a\ln(1-2\lambda\mu\nu)]\exp\left(\frac{\lambda\mu^2a^2}{1-2\lambda\mu\nu}\right). \end{aligned} \tag{5.2}$$

To illustrate the use of the similarity transformation (2.7), we investigate the Glauber P-representation (see [4]) of an operator $f(a, a^\dagger)$:

$$f(a,a^\dagger) = \int\frac{\mathrm{d}^2\alpha}{\pi}\, P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|. \tag{5.3}$$

Applying the similarity transformation to both sides of (5.3), we get

$$\begin{aligned} Wf(a,a^\dagger)W^- &= f(\mu a+\nu a^\dagger, \sigma a+\tau a^\dagger) \\ &= \int\frac{\mathrm{d}^2\alpha}{\pi}\, P(\alpha,\alpha^*)\, W|\alpha\rangle\langle\alpha|W^{-1}. \end{aligned}$$

Using IWOP, and expressions (4.6) and (4.7)

$$\begin{aligned} &f(\mu a+\nu a^\dagger,\sigma a+\tau a^\dagger) \\ &\quad = \int\frac{\mathrm{d}^2\alpha}{\pi}\, P(\alpha,\alpha^*)(\mu\tau)^{-1/2} \\ &\quad\quad \times :\exp\left(-|\alpha|^2+\frac{\alpha a^\dagger}{\mu}+\frac{\alpha^* a}{\tau}+\frac{\sigma\alpha^2}{2\mu}+\frac{\nu\alpha^{*2}}{2\tau}-\frac{\nu a^{\dagger 2}}{2\mu}-\frac{\sigma a^2}{2\tau}-a^\dagger a\right):. \end{aligned} \tag{5.4}$$

When $P(\alpha, \alpha^*)$ is known, performing the integration can produce the normal product form of the operator $f(\mu a + \nu a^\dagger, \sigma a + \tau a^\dagger)$. We now present a specific example.

Consider the P-representation of $\exp(-\lambda a^\dagger a)$ given by

$$\mathrm{e}^{-\lambda a^\dagger a} = \int\frac{\mathrm{d}^2\alpha}{\pi}\,\mathrm{e}^\lambda\exp[(1-\mathrm{e}^\lambda)|\alpha|^2]|\alpha\rangle\langle\alpha|. \tag{5.5}$$

Using (5.4), we obtain

$$\begin{aligned} &\mathrm{e}^{-\lambda(\sigma a+\tau a^\dagger)(\mu a+\nu a^\dagger)} \\ &\quad = \mathrm{e}^\lambda\int\frac{\mathrm{d}^2\alpha}{\pi}(\mu\tau)^{-1/2} \\ &\quad\quad \times :\exp\left(-\mathrm{e}^\lambda|\alpha|^2+\frac{\alpha a^\dagger}{\mu}+\frac{\alpha^* a}{\tau}+\frac{\sigma\alpha^2}{2\mu}+\frac{\nu\alpha^{*2}}{2\tau}-\frac{\nu a^{\dagger 2}}{2\mu}-\frac{\sigma a^2}{2\tau}-a^\dagger a\right): \\ &\quad = \mathrm{e}^\lambda R^{-1/2}\exp\left(\frac{\nu\tau(1-\mathrm{e}^{2\lambda})}{2R}a^{\dagger 2}\right)\exp[a^\dagger a(\lambda-\ln R)]\exp\left(\frac{\sigma\mu(1-\mathrm{e}^{2\lambda})}{2R}a^2\right) \end{aligned} \tag{5.6}$$

where $R \equiv \mu\tau\, e^{2\lambda} - \nu\sigma$. Rewriting (5.6) as

$$e^{-\lambda(\sigma a+\tau a^\dagger)(\mu a+\nu a^\dagger)} = e^{-\lambda(\mu\sigma a^2+\tau\nu a^{\dagger 2}+\nu\sigma a a^\dagger+\mu\tau a^\dagger a)}$$
$$= e^{-\lambda(ha^2+ga^{\dagger 2}+fa^\dagger a+\nu\sigma)}$$

where $h = \mu\sigma$, $g = \tau\nu$ and $f = 1 + 2\sigma\nu$ we recast this result in terms of f, g, and h to obtain the useful result

$$e^{-\lambda(ha^2+ga^{\dagger 2}+fa^\dagger a)}$$
$$= R^{-1/2} e^{(1/2)\lambda(1+f)} \exp\left(\frac{g(1-e^{2\lambda})}{2R}\, a^{\dagger 2}\right) \exp[a^\dagger a(\lambda - \ln R)]$$
$$\times \exp\left(\frac{h(1-e^{2\lambda})}{2R}\, a^2\right).$$

R may also be expressed in terms of f, g and h as $R = 2gh\, e^{2\lambda}/(f-1) - (f-1)/2$.

The results of the first example could also have been obtained using the P representation by noting that $e^{\lambda a^2} = \int \pi^{-1}\, d^2\alpha\, e^{\lambda\alpha^2}|\alpha\rangle\langle\alpha|$ and then using the methods above.

Conclusions

Although in quantum mechanics non-Hermitian operators such as a and $a^\dagger$ have found widespread use, the discussion of transformations has largely been restricted to unitary transformations. We have produced the normally ordered form of the general linear similarity transformation of a and $a^\dagger$ and shown that it presents some very useful properties. In particular, it allows us to choose, for any linearly transformed annihilation operator $d = \mu a + \nu a^\dagger$ having eigenkets $|\delta\rangle$, a whole range of associated creation operators $g^\dagger = \sigma a + \tau a^\dagger$ satisfying only $[d, g^\dagger] = 1$ whose eigenbras $\langle\gamma|$ may be substituted for those of $d^\dagger$ in the closure relation. The generality of this result suggests wide applicability.

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